Circular Coloring, Circular Flow, and Homomorphisms of Signed Graphs

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Publications and preprints

Circular colorings of signed graphs

[NWZ21] Circular chromatic number of signed graphs. With R. Naserasr and X. Zhu. *Electronic Journal of Combinatorics, 28(2)(2021), #P2.44, 40 pp.*

[KNNW21+] Circular $(4-\epsilon)$-coloring of signed graphs. With F. Kardos, J. Narboni, and R. Naserasr. *SIAM Journal on Discrete Mathematics.* Accepted subject to a minor revision.

[NW21] Circular coloring of signed bipartite planar graphs. With R. Naserasr. *In: Nešetřil J., Perarnau G., Rué J., Serra O. (eds) Extended Abstracts EuroComb 2021. Trends in Mathematics, vol 14. Birkhäuser, Cham.*

[NW21+] Signed bipartite circular cliques and a bipartite analogue of Grötzsch's theorem. With R. Naserasr. *Submitted.*

Circular flows in signed graphs

[LNNWZ22+] Circular flow in mono-directed signed graphs. With J. Li, R. Naserasr, and X. Zhu. *In preparation.*

- Strongly $\mathbb{Z}_2k$-connectedness and mapping signed bipartite planar graphs to cycles. With J. Li, Y. Shi, and C. Wei. *In preparation.*

Publications and preprints

Homomorphisms of signed graphs

[NPW22] *Density of C_{-4}-critical signed graphs.* With R. Naserasr and L.A. Pham. *Journal of Combinatorial Theory, Series B, 153 (2022), 81-104.*

[NSWX21+] *Mapping sparse signed graphs to* (K_{2k}, M)*.* With R. Naserasr, R. Škrekovski, and R. Xu. *Submitted.*

- *Bounding series-parallel graphs and cores of signed K_4-subdivisions.* With R. Naserasr. *Journal of Combinatorial Mathematics and Combinatorial Computing, (To the special volume for Gary MacGillivray), 116 (2021), 27-51.*

1. Introduction
   - Start from the 4-color theorem
   - Circular colorings and circular flows in signed graphs
   - Bipartite analog of Jaeger-Zhang conjecture

2. Circular chromatic numbers and circular flow indices of signed graphs
   - Bounding χ_c of some families
   - Bounding Φ_c of some families
   - Conclusion

The 4-color theorem

The 4-color theorem [AH76]

Every planar graph is 4-colorable.

4-color theorem restated

- [Wag37] Every graph with no K_5-minor is 4-colorable.
- [Tai80] Every cubic bridgeless planar graph is 3-edge-colorable.
- [Tut66] Every cubic bridgeless planar graph admits a nowhere-zero 4-flow.

From the 4-color theorem

- **Vertex-coloring theory and minor theory**

Hadwiger's conjecture [Had43]

Every graph with no K_{k-1}-minor is k-colorable.

- **Edge-coloring theory**

Seymour's edge-coloring conjecture [Sey75]

Every planar k-regular multigraph with no odd-cut of size less than k is k-edge-colorable.

- **Flow theory**

Tutte's 4-flow conjecture [Tut66]

Every bridgeless Petersen-minor-free graph admits a nowhere-zero 4-flow.

Signed graphs

- A signed graph (G, σ) is a graph G together with an assignment $\sigma : E(G) \to \{+, -\}$.
- The sign of a closed walk (especially, a cycle) is the product of signs of all the edges of it.
- A switching at a vertex v is to switch the signs of all the edges incident to this vertex.

Theorem [Zas82]

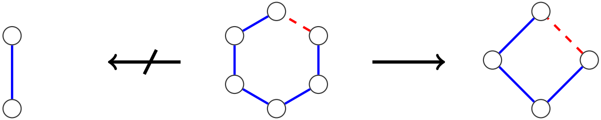
Signed graphs (G, σ) and (G, σ') are switching equivalent if and only if they have the same set of negative cycles.

Homomorphisms of signed graphs

- A homomorphism of (G, σ) to (H, π) is a mapping φ from $V(G)$ and $E(G)$ to $V(H)$ and $E(H)$ respectively, such that the adjacencies, the incidences and the signs of closed walks are preserved. When there exists one, we write $(G, \sigma) \to (H, \pi)$.
- A homomorphism of (G, σ) to (H, π) is said to be edge-sign preserving if, furthermore, it preserves the signs of the edges. When there exists one, we write $(G, \sigma) \xrightarrow{s.p.} (H, \pi)$.

Lemma [NSZ21]

$(G, \sigma) \to (H, \pi) \Leftrightarrow \exists \sigma' \equiv \sigma, (G, \sigma') \xrightarrow{s.p.} (H, \pi)$.



Necessary conditions for admitting homomorphisms

- Given $ij \in \mathbb{Z}_2^2$, the ij-girth of (G, σ), denoted $g_{ij}(G, \sigma)$, is the length of a shortest closed walk in (G, σ) satisfying that
  - the number of negative edges (counting multiplicity) is congruent to $i \pmod 2$;
  - the total number of edges (counting multiplicity) is congruent to $j \pmod 2$.

No-homomorphism lemma [NSZ21]

If $(G, \sigma) \to (H, \pi)$, then $g_{ij}(G, \sigma) \geq g_{ij}(H, \pi)$ for each $ij \in \mathbb{Z}_2^2$.

Generalizations from the 4-color problem

Odd Hadwiger's conjecture [GS90s]

Given a graph G, if $(G, -)$ has no $(K_{k-1}, -)$-minor, then G is k-colorable.

Signed projective cube conjecture [Gue05, Nas07]

Every signed planar graph (G, σ) satisfying that $g_{ij}(G, \sigma) \geq g_{ij}(C_{-k})$ for each $ij \in \mathbb{Z}_2^2$ admits a homomorphism to $SPC(k-1)$.

Jaeger-Zhang conjecture [Zha02]

Every planar graph of odd-girth at least $4k+1$ admits a circular $\frac{2k+1}{k}$-coloring.

Jaeger's circular flow conjecture

Tutte's 3-flow conjecture [Tut66]

Every 4-edge-connected graph admits a nowhere-zero 3-flow.

Jaeger's circular flow problem [Jae84]

Every $4k$-edge-connected graph admits a circular $\frac{2k+1}{k}$-flow.

- It has been disproved for $k \geq 3$ [HLWZ18];
- It has been verified that the $6k$-edge-connectivity is a sufficient condition for a graph to admit a circular $\frac{2k+1}{k}$-flow [LTWZ13].

Duality: circular flow and circular coloring

Let p and q be two positive integers satisfying $p \geq 2q$.

A circular $\frac{p}{q}$-flow in a graph G is a pair (D, f) where D is an orientation on G and $f : E(G) \to \mathbb{Z}$ satisfying that $q \leq |f(e)| \leq p - q$ and for each vertex v,

$$\sum_{(v,w)\in D} f(vw) - \sum_{(u,v)\in D} f(uv) = 0.$$

A circular $\frac{p}{q}$-coloring of a graph G is a mapping $\varphi : V(G) \to \{1, 2, ..., p\}$ such that $q \leq |\varphi(u) - \varphi(v)| \leq p - q$ for each edge $uv \in E(G)$.

Lemma [GTZ98]

A plane graph G admits a circular $\frac{p}{q}$-coloring if and only if its dual graph G^* admits a circular $\frac{p}{q}$-flow.

Jaeger-Zhang Conjecture

Jaeger-Zhang Conjecture [Zha02]

Every planar graph of odd-girth at least $4k+1$ admits a circular $\frac{2k+1}{k}$-coloring.

- $k = 1$: Grötzsch's theorem;
- $k = 2$: verified for odd-girth 11 [DP17; CL20];
- $k = 3$: verified for odd-girth 17 [CL20; PS22];
- $k \geq 4$:
  - verified for odd-girth $8k-3$ [Zhu01];
  - verified for odd-girth $\frac{20k-2}{3}$ [BKKW02];
  - verified for odd-girth $6k+1$ [LTWZ13].

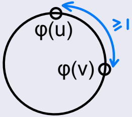
Circular coloring of signed graphs

Let C^r be a circle of circumference r.

Definition [NWZ21]

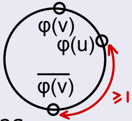
Given a signed graph (G, σ) with no positive loop and a real number r, a circular r-coloring of (G, σ) is a mapping $\varphi : V(G) \to C^r$ such that for each positive edge uv of (G, σ),

$$d_{C^r}(\varphi(u), \varphi(v)) \geq 1,$$



and for each negative edge uv of (G, σ),

$$d_{C^r}(\varphi(u), \overline{\varphi(v)}) \geq 1.$$



The circular chromatic number of (G, σ) is defined as

$$\chi_c(G, \sigma) = \inf\{r \geq 1 : (G, \sigma) \text{ admits a circular } r\text{-coloring}\}.$$

Circular $\frac{p}{q}$-coloring of signed graphs

For $i, j, x \in \{0, 1, ..., p-1\}$,

$$d_{(\mathrm{mod}\ p)}(i, j) = \min\{|i-j|, p-|i-j|\} \text{ and } \bar{x} = x + \frac{p}{2} (\mathrm{mod}\ p).$$

Given a positive even integer p and a positive integer q satisfying $q \leq \frac{p}{2}$, a circular $\frac{p}{q}$-coloring of a signed graph (G, σ) is a mapping $\varphi : V(G) \to \{0, 1, ..., p-1\}$ such that for any positive edge uv,

$$q \leq |\varphi(u) - \varphi(v)| \leq p - q,$$

and for any negative edge uv,

$$|\varphi(u) - \varphi(v)| \leq \frac{p}{2} - q \text{ or } |\varphi(u) - \varphi(v)| \geq \frac{p}{2} + q.$$

Signed circular cliques

Lemma [NWZ21]

A signed graph (G, σ) admits a circular $\frac{p}{q}$-coloring if and only if it admits an edge-sign preserving homomorphism to the signed circular clique $K^s_{p;q}$.

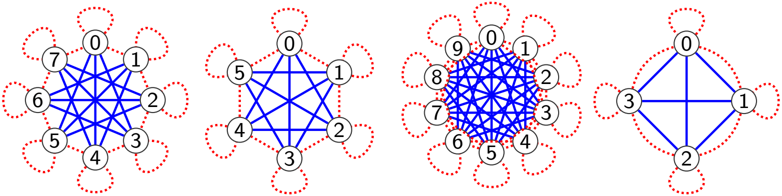


Figure: $K^s_{8;3} \prec K^s_{6;2} \prec K^s_{10;3} \prec K^s_{4;1}$

Orientation on signed graphs

- A bi-directed signed graph is a signed graph where each edge is assigned with two directions at both of its ends such that
  - in a positive edge, the ends are both directed from one endpoint to the other,
  - in a negative edge, either both ends are directed outward or both are directed inward.
- A mono-directed signed graph is a signed graph where each edge is assigned with one direction.

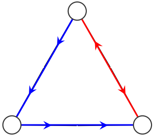


Figure: A bi-directed signed K_3

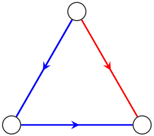


Figure: A mono-directed signed K_3

Circular $\frac{p}{q}$-flow in mono-directed signed graphs

Definition [LNWZ22+]

Given a positive even integer p and a positive integer q where $q \leq \frac{p}{2}$, a circular $\frac{p}{q}$-flow in a signed graph (G, σ) is a pair (D, f) where D is an orientation on G and $f : E(G) \to \mathbb{Z}$ satisfies the following conditions.

- For each positive edge e, $|f(e)| \in \{q, ..., p-q\}$;
- For each negative edge e, $|f(e)| \in \{0, ..., \frac{p}{2} - q\} \cup \{\frac{p}{2} + q, ..., p-1\}$;
- For each vertex v of (G, σ), $\sum_{(v,w) \in D} f(vw) = \sum_{(u,v) \in D} f(uv)$.

The circular flow index of (G, σ) is defined to be

$$\Phi_c(G, \sigma) = \min\{\frac{p}{q} \mid (G, \sigma) \text{ admits a circular } \frac{p}{q}\text{-flow}\}.$$

Circular $\frac{2\ell}{\ell-1}$-coloring and circular $\frac{2\ell}{\ell-1}$-flow

Lemma [LNWZ22+]

A signed plane graph (G,σ) admits a circular $\frac{p}{q}$-coloring if and only if its dual signed graph (G^*,σ^*) admits a circular $\frac{p}{q}$-flow, i.e.,

$$\chi_c(G,\sigma) \leq \frac{p}{q} \iff \Phi_c(G^*,\sigma^*) \leq \frac{p}{q}.$$

Let k be a positive integer.

- A signed graph $(G,+)$ admits a circular $\frac{2k+1}{k}$-coloring if and only if $(G,+) \to C_{2k+1}$.
- [NW21+] A signed bipartite graph (G,σ) admits a circular $\frac{4k}{2k-1}$-coloring if and only if $(G,\sigma) \to C_{-2k}$.

Homomorphisms of signed bipartite graphs

Proposition [HN90, NPW**22]**

For any integer k, a graph G is k-colorable if and only if $T_{k-2}(G)$ admits a homomorphism to C_{-k}.

4-color theorem restated

Given a planar graph G, the signed bipartite graph $T_2(G)$ admits a homomorphism to C_{-4}.

Lemma

- [NRS15] $G \to H \Leftrightarrow S(G) \to S(H)$;
- [N**W**Z21] Given a graph G, we have $\chi_c(S(G)) = 4 - \dfrac{4}{\chi_c(G)+1}$;
- [NRS15] $\chi(G) \leq k \Leftrightarrow S(G) \to (K_{k,k}, M)$ for $k \geq 3$.

Homomorphisms of signed bipartite graphs

4-color theorem restated [KNNW21+, NW21+]

Let G be a planar graph.

- $S(G) \to S(K_4)$;
- $S(G) \to \hat{B}^s_{16;5}$;
- $S(G) \to (K_{4,4}, M)$.

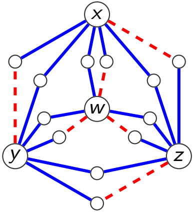


Figure: $S(K_4)$

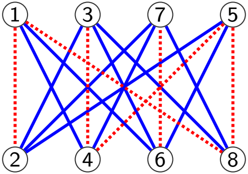


Figure: $\hat{B}^s_{16;5}$

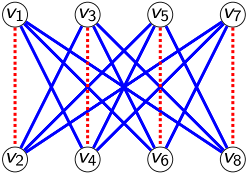


Figure: $(K_{4,4}, M)$

Signed bipartite analog of Jaeger-Zhang conjecture

Signed bipartite analog of Jaeger's circular flow conjecture

Every $g(k)$-edge-connected Eulerian signed graph admits a circular $\frac{4k}{2k-1}$-flow.

Signed bipartite analog of Jaeger-Zhang conjecture [NRS15]

Every signed bipartite planar graph of negative-girth at least $f(k)$ admits a homomorphism to C_{-2k}.

- $k = 2$: verified for negative-girth 8 (best possible) [NP**W**22];
- $k = 3$: verified for negative-girth 14 [LS**W**W22+];
- $k = 4$: verified for negative-girth 20 [LS**W**W22+];
- $k \geq 5$:
  - verified for negative-girth $8k - 2$ [CNS20];
  - verified for negative-girth $6k - 2$ [LN**W**Z22+].

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Bounding χ_c of some families

Let $\mathcal{C}$ be a class of signed graphs.

$$\chi_c(\mathcal{C}) = \sup\{\chi_c(G,\sigma) \mid (G,\sigma) \in \mathcal{C}\}.$$

$\mathcal{SD}_d$: the class of signed d-degenerate simple graphs

- For any positive integer $d \geq 2$, $\chi_c(\mathcal{SD}_d) = 2\lfloor \frac{d}{2} \rfloor + 2$ [N**W**Z21];

Theorem [KNNW**21+]**

If (G,σ) is a signed 2-degenerate simple graph on n vertices, then

$$\chi_c(G,\sigma) \leq 4 - \frac{2}{\lfloor \frac{n+1}{2} \rfloor}.$$

Moreover, this upper bound is tight for each value of $n \geq 2$.

The families $\mathcal{SSP}_k$ and $\mathcal{SK}$

$\mathcal{SSP}_k$: the class of signed series-parallel graphs of girth at least k

- $\chi_c(\mathcal{SSP}_3) = \frac{10}{3}$ [NWZ21]; $\chi_c(\mathcal{SSP}_4) = 3$.

$\mathcal{SK}$: the class of signed simple graphs whose underlying graph is k-colorable

- $\chi_c(\mathcal{SK}) = 2k$.

Theorem [NWZ21]

For any integers $k, g \geq 2$ and any $\epsilon > 0$, there is a graph G of girth at least g satisfying that $\chi(G) = k$ and a signature σ such that $\chi_c(G, \sigma) > 2k - \epsilon$.

The families $\mathcal{SP}_k$

$\mathcal{SP}_k$: the class of signed planar graphs of girth at least k

- $\frac{14}{3} \leq \chi_c(\mathcal{SP}_3) \leq 6$ [N**W**Z21];

Theorem [KN21]

There is a signed planar graph which is not $\{\pm 1, \pm 2\}$-colorable.

- $\chi_c(\mathcal{SP}_7) \leq 3$.

Theorem [NSW**X21+]**

Every signed graph with $mad(G) < \frac{14}{5}$ admits a homomorphism to (K_6, M). Moreover, the bound $\frac{14}{5}$ is the best possible.

$\chi_c(\mathcal{SBP}_4) \cong 4$

$\mathcal{SBP}_\ell$: the class of signed bipartite planar graphs of negative-girth at least ℓ

Theorem [KNNW**21+]**

If (G, σ) is a signed bipartite planar simple graph on n vertices, then

$$\chi_c(G, \sigma) \leq 4 - \frac{4}{\lfloor \frac{n+2}{2} \rfloor}.$$

Moreover, this upper bound is tight for each value of $n \geq 2$.

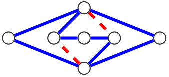

Figure: Γ_3

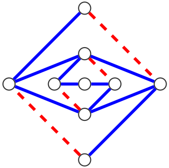

Figure: Γ_4

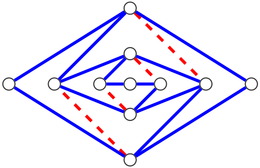

Figure: Γ_5

$\frac{14}{5} \leq \chi_c(\mathcal{SBP}_6) \leq 3$

Theorem [NW21+]

For $(G, \sigma) \in \mathcal{SBP}_6$, $(G, \sigma) \to (K_{3,3}, M)$.

Theorem [DKK16; NRS13]

Every signed planar graph (G, σ) satisfying that $g_{ij}(G, \sigma) \geq g_{ij}(SPC(5))$ for each $ij \in \mathbb{Z}_2^2$ admits a homomorphism to $SPC(5)$.

Key lemma [NRS13]

For $(G, \sigma) \in \mathcal{SBP}_6$, there are 6 disjoint subsets of edges, $E_1, E_2, ..., E_6$, such that each of the signed graphs (G, σ_i), $i \in [6]$, where E_i is the set of negative edges of (G, σ_i), is switching equivalent to (G, σ).

$\chi_c(\mathcal{SBP}_8) \cong \frac{8}{3}$

Theorem [NPW22]

If $\hat{G}$ is a C_{-4}-critical signed graph that is not isomorphic to $\hat{W}$, then
$$|E(G)| \geq \frac{4|V(G)|}{3}.$$

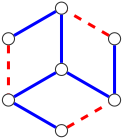

Figure: C_{-4}-critical signed graph $\hat{W}$

Corollary [NPW22]

For $(G, \sigma) \in \mathcal{SBP}_8$, $(G, \sigma) \to C_{-4}$.

Bounding Φ_c of some families

Conjectures [LNWZ22+]

- Every 3-edge-connected signed graph admits a circular 5-flow.
- Every 5-edge-connected signed graph admits a circular 3-flow.
- Every 2-edge-connected signed Petersen-minor-free graph admits a circular 8-flow.

Connection to the $\mathbb{Z}_{2k}$-connectivity

- Every 3-edge-connected signed graph admits a circular 6-flow.
- Every 4-edge-connected signed graph admits a circular 4-flow.

Via the Nash-Williams and Tutte theorem

- For every 6-edge-connected signed graph (G, σ), $\Phi_c(G, \sigma) < 4$.

Highly edge-connected signed graphs

Theorem [LNWZ22+]

Given a signed graph (G, σ), we have the following claims:

1. If G is $(6k-1)$-edge-connected, then $\Phi_c(G, \sigma) \leq \frac{4k}{2k-1}$.
2. If G is $6k$-edge-connected, then $\Phi_c(G, \sigma) < \frac{4k}{2k-1}$.
3. If G is $(6k+1)$-edge-connected, then $\Phi_c(G, \sigma) \leq \frac{8k+2}{4k-1}$.
4. If G is $(6k+2)$-edge-connected, then $\Phi_c(G, \sigma) \leq \frac{2k+1}{k}$.
5. If G is $(6k+3)$-edge-connected, then $\Phi_c(G, \sigma) < \frac{2k+1}{k}$.
6. If G is $(6k+4)$-edge-connected, then $\Phi_c(G, \sigma) \leq \frac{8k+6}{4k+1}$.

Highly edge-connected signed Eulerian graphs

Theorem [LNWZ22+]

Every $(6k-2)$-edge-connected Eulerian signed graph admits a circular $\frac{4k}{2k-1}$-flow.

Theorem [LNWZ22+]

Every signed bipartite planar graph of negative-girth at least $6k-2$ admits a homomorphism to C_{-2k}.

The class $\mathcal{SP}_k^*$

$\mathcal{SP}_k^*$: the class of signed planar graphs (G, σ) satisfying that $g_{ij}(G, -\sigma) \geq g_{ij}(C_{-k})$ for each $ij \in \mathbb{Z}_2^2$.

- $\mathcal{SP}_{2\ell+1}^*$ consists of all signed planar graphs $(G, +)$ of odd-girth at least $2\ell + 1$.
- $\mathcal{SP}_{2\ell}^*$ consists of all signed bipartite planar graphs of negative-girth at least 2ℓ, i.e., $\mathcal{SBP}_{2\ell}$.

Circular chromatic number of $\mathcal{SP}_k^*$ and $\mathcal{SP}_k$

k	Bounds on $\chi_c(\mathcal{SP}_k^*)$	Reference	Bounds on $\chi_c(\mathcal{SP}_k)$	Reference
2	$\chi_c(\mathcal{SP}_2^*) = 4$	Prop. 3.1.7	$\chi_c(\mathcal{SP}_2) = 8$	the 4-color thm
3	$\chi_c(\mathcal{SP}_3^*) = 4$	the 4-color thm	$\frac{14}{3} \leqslant \chi_c(\mathcal{SP}_3) \leqslant 6$	Thm. 4.4.1
4	$\chi_c(\mathcal{SP}_4^*) \cong 4$	Thm. 7.2.1	$\chi_c(\mathcal{SP}_4) \leqslant 4$	[MRŠ16]
5	$\chi_c(\mathcal{SP}_5^*) = 3$	[Grö58], [SY89]	$*$	
6	$\frac{14}{5} \leqslant \chi_c(\mathcal{SP}_6^*) \leqslant 3$	Thm. 9.2.4	$*$	
7	$*$		$\chi_c(\mathcal{SP}_7) \leqslant 3$	Cor. 10.5.5
8	$\chi_c(\mathcal{SP}_8^*) \cong \frac{8}{3}$	Thm. 8.5.3	$*$	
10	$*$		$\chi_c(\mathcal{SP}_{10}) \leqslant \frac{8}{3}$	[LS**W**W22+]
11	$\chi_c(\mathcal{SP}_{11}^*) \leqslant \frac{5}{2}$	[DP17], [CL20]	$*$	
14	$\chi_c(\mathcal{SP}_{14}^*) \leqslant \frac{12}{5}$	[LS**W**W22+]	$*$	
17	$\chi_c(\mathcal{SP}_{17}^*) \leqslant \frac{7}{3}$	[CL20], [PS22]	$*$	
20	$\chi_c(\mathcal{SP}_{20}^*) \leqslant \frac{16}{7}$	[LS**W**W22+]	$*$	
...	...	...	...	
$6p-2$	$\chi_c(\mathcal{SP}_{6p-2}^*) \leqslant \frac{4p}{2p-1}$	Thm. 6.3.9	$\chi_c(\mathcal{SP}_{6p-2}) \leqslant \frac{8p-2}{4p-3}$	Thm. 6.3.5
$6p-1$	$\chi_c(\mathcal{SP}_{6p-1}^*) \leqslant \frac{4p}{2p-1}$	[LWZ20]	$\chi_c(\mathcal{SP}_{6p-1}) \leqslant \frac{4p}{2p-1}$	Thm. 6.3.5
$6p$	$*$		$*$	
$6p+1$	$\chi_c(\mathcal{SP}_{6p+1}^*) \leqslant \frac{2p+1}{p}$	[LTWZ13]	$\chi_c(\mathcal{SP}_{6p+1}) \leqslant \frac{8p+2}{4p-1}$	Thm. 6.3.5
$6p+2$	$*$		$\chi_c(\mathcal{SP}_{6p+2}) \leqslant \frac{2p+1}{p}$	Thm. 6.3.5

Jaeger-Zhang conjecture and some extensions

Conjecture

Given a positive integer k, we have that

$$\chi_c(\mathcal{SP}^*_{4k+1}) = \chi_c(\mathcal{SP}^*_{4k+2}) = \frac{2k+1}{k},$$

and

$$\chi_c(\mathcal{SP}^*_{4k-1}) = \chi_c(\mathcal{SP}^*_{4k}) = \frac{4k}{2k-1}.$$

Thanks for your attention!